

XVII. *On the Method of correspondent Values, &c.* By Edward Waring, M. D. F. R. S. and Lucaian Professor of the Mathematics at Cambridge.

Read May 28, 1789.

I.

1. **I**N the year 1762 I published a method of finding when two roots of a given equation $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$ are equal, by finding the common divisors of the two quantities $a^n - pa^{n-1} + qa^{n-2} - \&c.$, and $na^{n-1} - \overline{n-1}pa^{n-2} + \overline{n-2}qa^{n-3} - \&c.$, and observed if they admitted only one simple divisor $(a - A)$, then two roots were only equal; if a quadratic $(a^2 - Aa + B)$, then two roots of the equation became twice equal; if a cubic $(a^3 - Aa^2 + Ba - C)$, then two roots became thrice equal; and so on: or, to express in more general terms what follows from the same principles, if the common divisor be $\overline{a-b'} \times \overline{a-c'} \times \overline{a-d'} \times \&c.$, then $r+1$ roots of the given equation will be b , $s+1$ roots will be c , $t+1$ will be d , &c.; and it immediately follows, from the principles delivered in the second edition of the same Book, published in 1770, that to find when $r+1$, $s+1$, $t+1$, &c. roots are respectively equal requires $r+s+t$, &c. equations of condition, which are deducible from the well known method of finding the common divisors of two quantities in this case of $a^n - pa^{n-1} + qa^{n-2} - \&c.$, $na^{n-1} - \overline{n-1}pa^{n-2} + \overline{n-2}qa^{n-3} - \&c.$ of the terms of their remainders, &c.

In

In the book above mentioned the equations of condition are given, which discover when two roots are equal in the equations $x^3 - px^2 + qx - r = 0$, $x^4 + qx^3 - rx^2 + sx - t = 0$, in the two latter equations the second term is wanting, which may easily be exterminated; but it may as easily be restored by substituting for q, r, s , &c. in the equation of condition found the quantities resulting from the common transformation of equations to destroy the second term,

2. Another rule contained in the same Book is the substitution of the roots of the equation $na^{n-1} - \overline{n-1}pa^{n-2} + \overline{n-2}qa^{n-3} - \&c. = 0$ respectively for a in the quantity $a^n - pa^{n-1} + qa^{n-2} - \&c.$, and multiplication of all the quantities resulting into each other; their content will give the equation of condition, when two roots are equal.

Mr. HUDDE first discovered, that if the successive terms of the given equation are multiplied into an arithmetical series, the resulting equation will contain one of any two equal roots, and m of the $m+1$ equal roots in the given equation.

3. If 3, 4, 5, . . . r roots of the equation are equal, find a common divisor of 3, 4, 5, . . . r of the subsequent quantities $a^n - pa^{n-1} + qa^{n-2} - \&c.$, $na^{n-1} - \overline{n-1}pa^{n-2} + \overline{n-2}qa^{n-3} - \&c.$, $n \cdot \overline{n-1}a^{n-2} - \overline{n-1} \cdot \overline{n-2}pa^{n-3} + \overline{n-2} \cdot \overline{n-3}qa^{n-4} - \overline{n-3} \cdot \overline{n-4}ra^{n-5} + \&c.$, $n \cdot \overline{n-1} \cdot \overline{n-2}a^{n-3} - \overline{n-1} \cdot \overline{n-2} \cdot \overline{n-3}pa^{n-4} + \overline{n-2} \cdot \overline{n-3} \cdot \overline{n-4}qa^{n-5} - \&c.$, . . . $n \cdot \overline{n-1} \cdot \overline{n-2} \cdot \overline{n-3} \cdot \overline{n-4}ra^{n-5} + \&c.$; which will probably be best done by dividing all the preceding quantities by the quantity of the least dimension of a , and the divisor and all the remainders by that quantity which has the least dimensions amongst them; and so on: there will result 2, 3, 4, . . . $r-1$ equations of condition; and in this case it is

observed, in the before-mentioned Book, that (if the common divisor be $(a - A)$) it will once only admit of 3, 4, 5, . . . r equal roots; if it be a quadratic, then it will twice admit of those equal roots; and so on.

4. If the roots of the equation of the least dimensions be substituted for a in the remaining equations, and each of the resulting values of the same equation be multiplied into each other, there will result the $r - 1$ equations of condition: and the same may be deduced also from the several equations conjointly.

The equations of conditions found by the first method, if the divisions were not properly instituted, may admit of more rational divisors than necessary, of which some are the equations of conditions required.

2.

1. In the year 1776, I published in the *Meditationes Analyticæ* a new method of differences for the resolution of the following problem.

Given the sums of a swiftly converging series $ax + bx^2 + cx^3 + dx^4 + \&c.$, when the values of x are respectively $\pi, \rho, \varsigma, \&c.$; to find the sum of the series when x is τ , that is, given $S\pi = a\pi + b\pi^2 + c\pi^3 + d\pi^4 + \&c.$, $S\rho = a\rho + b\rho^2 + c\rho^3 + \&c.$, $S\sigma = a\sigma + b\sigma^2 + c\sigma^3 + \&c. \&c.$; to find $S\tau = a\tau + b\tau^2 + c\tau^3 + \&c.$

To resolve this problem I multiplied the quantities, $S\pi, S\rho, S\sigma, \&c.$ respectively into unknown co-efficients $\alpha, \beta, \gamma, \&c.$ and there resulted

$$\alpha\pi a + \alpha\pi^2 b + \alpha\pi^3 c + \&c.$$

$$\beta\rho a + \beta\rho^2 b + \beta\rho^3 c + \&c.$$

$$\gamma\sigma a + \gamma\sigma^2 b + \gamma\sigma^3 c + \&c.$$

$$\&c. \quad \&c. \quad \&c.$$

and then made the sum of each of the terms respectively equal to its correspondent term of the quantity $\tau a + \tau^2 b + \tau^3 c + \&c.$, and consequently $\alpha\pi + \beta\rho + \gamma\sigma + \&c. = \tau$, $\alpha\pi^2 + \beta\rho^2 + \gamma\sigma^2 + \&c. = \tau^2$, $\alpha\pi^3 + \beta\rho^3 + \gamma\sigma^3 + \&c. = \tau^3$, &c. I assumed as many equations of this kind as there were given values π, ρ, σ , &c. of x ; and consequently as many equations resulted as unknown quantities α, β, γ , &c.; whence, by the common resolution of simple equations, or more easily from differences, can be found the unknown quantities α, β, γ , &c., and thence the equation sought $\alpha \times S\pi + \beta \times S\rho + \gamma \times S\sigma + \&c. = S\tau$ nearly.

3. In the Meditationes are assumed for π, ρ, σ , &c. the quantities $p, 2p, 3p, 4p, \dots \overline{n-2p}, \overline{n-1p}$, and np for τ ; which, if substituted for their values in the preceding equations, will give $\alpha + 2\beta + 3\gamma + 4\delta + \&c. = n$, $\alpha + 4\beta + 9\gamma + 16\delta + \&c. = n^2$, $\alpha + 8\beta + 27\gamma + \&c. = n^3$, $\alpha + 16\beta + 81\gamma + \&c. = n^4$; and if the sums of the series $ax + bx^2 + cx^3 + \&c.$ which respectively correspond to the values $p, 2p, 3p, \dots \overline{n-1p}$ of x be $S_1, S_2, S_3, S_4, \dots S_{n-1}$, and the sum of the series $ax + bx^2 + cx^3 + \&c.$ which corresponds to n value of x be S_n ; then will $S_n = nS_{n-1} - n \cdot \frac{n-1}{2} S_{n-2} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{2} S_{n-3} \dots \pm n S_1$ nearly, which equation is given in the above-mentioned Book.

3. The logarithm from the number, the arc from the sine, &c. are found by serieses of the formula $ax + bx^2 + cx^3 + \&c.$; and consequently this equation is applicable to them.

4. In the same Book is assumed a series $ax^r + bx^{r+s} + cx^{r+2s} + dx^{r+3s} + \&c.$ of a more general formula than the preceding, and in it for x substituted $\alpha, \beta, \gamma, \delta, \&c., m$; and $S\alpha, S\beta, S\gamma, S\delta, \&c.$; S_m for the resulting sums, and thence deduced

$$S_m = \frac{m^r \times m^s - \beta^s \cdot m^s - \gamma^s \cdot m^s - \delta^s \cdot \&c.}{\alpha^r \times \alpha^s - \beta^s \cdot \alpha^s - \gamma^s \cdot \alpha^s - \delta^s \cdot \&c.} \times S\alpha + \frac{m^r \times m^s - \alpha^s \cdot m^s - \gamma^s \cdot m^s - \delta^s \cdot \&c.}{\beta^r \times \beta^s - \alpha^s \cdot \beta^s - \gamma^s \cdot \beta^s - \delta^s \cdot \&c.} \\ \times S\beta + \frac{m^r \times m^s - \alpha^s \cdot m^s - \beta^s \cdot m^s - \delta^s \cdot \&c.}{\gamma^r \times \gamma^s - \alpha^s \cdot \gamma^s - \beta^s \cdot \gamma^s - \delta^s \cdot \&c.} \times S\gamma + \frac{m^r \times m^s - \alpha^s \cdot m^s - \beta^s \cdot m^s - \gamma^s \cdot \&c.}{\delta^r \times \delta^s - \alpha^s \cdot \delta^s - \beta^s \cdot \delta^s - \gamma^s \cdot \&c.} \\ \times S\delta + \&c. \text{ nearly.}$$

Cor. If for r and s be assumed respectively 1, the series becomes $ax + bx^2 + cx^3 + \&c.$ of the same formula as the preceding: if $r=0$ and $s=1$, the series becomes $a + bx + cx^2 + \&c.$ The latter case will be the same as the former, when one of the quantities (α) substituted for x and its correspondent sum $S\alpha$, both become $=0$, and the equation deduced in both cases the same.

5. If $\pi, \rho, \sigma, \&c.$ respectively denote $r, r+p, r+2p, \dots r+n-2p, r+n-1p$, and $\tau = r+np$; and $S, S_1, S_2, S_3, \dots S_{n-2}, S_{n-1}$, be the sums either resulting from the series $ax + bx^2 + cx^3 + \&c.$ or the series $A + ax + bx^2 + cx^3 + \&c.$, which respectively correspond to the values $r, r+p, r+2p, \&c.$ of x ; and S_n the sum of the same series which corresponds to the value $r+np$ of x ; then will $S_n = nS_{n-1} - n \cdot \frac{n-1}{2} S_{n-2} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} S_{n-3} - \dots \pm n \cdot \frac{n-1}{2} S_2 \mp nS_1 \pm S$ nearly; this equation differs from the preceding by the last term S not vanishing; in the preceding case S became $=0$, for it was the sum of the series $ax + bx^2 + cx^3 + \&c.$, which corresponded to $x=0$.

6. From

6. From the Meditationes it appears, that $r^m - n \times \overline{r \pm p^m} + n \cdot \frac{n-1}{2} \overline{r \pm 2p^m} - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \overline{r \pm 3p^m} + \&c.$ to the end of the series = 0, if m is less than n , and m and n are whole numbers; but if $m = n$, then it will = $\pm 1 \cdot 2 \cdot 3 \cdot 4 \dots \overline{n-1} \cdot np^m$; whence it is manifest, that for the n first terms of the series $A + ax + bx^2 + \&c.$ the equations are true; and for the $n-1$ first terms of the series $ax + bx^2 + cx^3 + \&c.$ and in the successive term of both the serieses they will err by a quantity nearly = $\pm 1 \cdot 2 \cdot 3 \dots n \times p^n \times r^{-n} \times$ co-efficient of the term; and the errors of every subsequent term (x^{4+n}) will be nearly as $\pm m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} \dots \frac{m-b+1}{b} \times p^n \times r^{-n} \times$ co-efficient of the term x^{b+n} , if for $r, r+p, r+2p, \&c.$ be substituted $1, 1 + \frac{p}{r}, 1 + \frac{2p}{r}, \&c.$

7. Let the preceding equation $S_n = n \overline{S_{n-1}} - n \cdot \frac{n-1}{2} \overline{S_{n-2}} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \overline{S_{n-3}} - \&c. = n \times \log. \overline{r-p} - n \cdot \frac{n-1}{2} \log. \overline{r-2p} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \log. \overline{r-3p} + \&c. = \log. \frac{r \times \overline{r-2p'} \times \overline{r-4p'} \times \overline{r-6p'} \times \&c.}{\overline{r-p'} \times \overline{r-3p'} \times \overline{r-5p'} \times \&c.} = \log. K$, where $s, s', s'', \&c.$ denote the co-efficients of the alternate terms of the binomial theorem, viz. $s = n \cdot \frac{n-1}{2}, s' = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}, \&c.$, and $t = n, t' = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}, \&c.$ the co-efficients of the remaining alternate terms; the numerator $r \times \overline{r-2p'} \times \overline{r-4p'} \times \overline{r-6p'} \times \&c. =$ (if $N = 2^{n-1}$) $r^N - Pp r^{N-1} + Qp^2 r^{N-2} - Rp^3 r^{N-3} \dots Lp^{n-1} \times r^{N-n+1} \pm Mp^n r^{N-n} \pm \&c.$; and the denominator $\overline{r-p'} \times \overline{r-3p'} \times \overline{r-5p'} \times \&c. = r^N - Pp r^{N-1} + Qp^2 r^{N-2} -$

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Rp

$Rp^3r^{n-3} + \dots Lp^{n-1}r^{N-n+1}(\pm M + 1 \cdot 2 \cdot 3 \dots \overline{n-1}) p^n r^{N-n} =$
 &c., whence the numerator and denominator have the n first
 terms the same, and the next succeeding terms differ by
 $1 \cdot 2 \cdot 3 \dots \overline{n-1} p^n r^{N-n}$; the numerator divided by the denomi-
 nator $= 1 \pm \frac{1 \cdot 2 \cdot 3 \dots \overline{n-1}}{r^n} p^n$ nearly, if r be a great number in
 proportion to p , &c. it would be $+$ when n is an odd number,
 and $-$ when even.

8. The logarithm of the fraction K by the common series
 $= K - 1 - \frac{\overline{K-1}^2}{2} + \frac{\overline{K-1}^3}{3} - \&c.$ has for its first term $= \pm$
 $\frac{1 \cdot 2 \cdot 3 \dots \overline{n-1}}{r^n} \times p^n$ nearly; for its second term the square of
 the first divided by 2, &c.

9. The error of this equation not only depends on the loga-
 rithm of K , which may be calculated to any degree of exact-
 ness, but in the calculus on the errors of the given loga-
 rithms.

10. If r be increased or diminished by any given number,
 the n first terms of the numerator and denominator will still
 result the same, and the next succeeding terms will differ by
 $1 \cdot 2 \cdot 3 \cdot 4 \dots n-1 \times p^n \times r^{N-n}$.

11. Let $n \cdot \frac{n-1}{2}$ numbers be 2, $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}$ num-
 bers be 4, $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} \cdot \frac{n-5}{6}$ numbers be 6, &c.;
 their sum, the sum of the products of every two, the con-
 tents of every three, four, five, &c. to $n-1$ of them will be
 equal to the sum, the sum of the products of every two, of
 the contents of every three, four, five, &c. to $\overline{n-1}$ of the
 following numbers, viz. n numbers which are 1, $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$
 numbers

numbers which are 3, $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5}$, which are 5, &c.; and the sum of the contents of every n of the former will be less than the sum of the contents of every n latter numbers by $1 \cdot 2 \cdot 3 \cdot 4 \dots \overline{n-1}$.

12. The method given in Art. 4. which I name a method of correspondent values, easily deduces and demonstrates the preceding equations, which cannot, without much difficulty, be done by the preceding method of differences; the method of correspondent values is much preferable to the method of differences, both for the facility of its deduction, and the generality of its resolution: for instance, from this method very easily can be deduced, &c. the subsequent and other similar equations.

Ex. 1. $S_n = n \overline{S_{n-1}} - n \cdot \frac{n-1}{2} \overline{S_{n-2}} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \overline{S_{n-3}} \dots$
 $\approx n S_1 \approx S$ nearly.

Ex. 2. $S_n + m = \frac{\overline{m+n} \cdot \overline{m+n-1} \cdot \overline{m+n-2} \dots \overline{m+2}}{1 \cdot 3 \dots n-1} \times \overline{S_{n-1}} -$
 $\frac{n-1}{1} \times A \times \frac{m+1}{m+2} \times \overline{S_{n-2}} + \frac{n-2}{2} \times B \times \frac{m+2}{m+3} \overline{S_{n-3}} - \frac{n-3}{3} \times C \times$
 $\frac{m+3}{m+4} \times \overline{S_{n-4}} + \frac{n-4}{4} \times D \times \frac{m+4}{m+5} \times \overline{S_{n-5}} - \&c.$ nearly, where the letters A, B, C, D, &c. denote the preceding co-efficients, and the converging series is the same as in the preceding example.

Ex. 3. Let the converging series be of the formula $ax + bx^3 + cx^5 + dx^7 + \&c.$; then will $S_n = \overline{2n-2} \overline{S_{n-1}} - \overline{2n-1} \times$
 $\frac{2n-4}{2} \overline{S_{n-2}} + \overline{2n-1} \times \frac{2n-2}{2} \times \frac{2n-6}{3} \overline{S_{n-3}} - \overline{2n-1} \cdot \frac{2n-2}{2} \cdot$
 $\frac{2n-3}{2} \times \frac{2n-8}{4} \overline{S_{n-4}} + \&c.$ nearly, of which the general term is
 $\overline{2n-1} \cdot \frac{2n-2}{2} \cdot \frac{2n-3}{3} \dots \frac{2n-l+1}{l-1} \times \frac{2n-2l}{l} \times \overline{S_{n-l}}.$

Ex. 4. Let the series be of the formula $A + ax^2 + bx^4 + cx^6 + \&c.$; then will $S_n = \frac{n-1}{n} \times \overline{2n-2} \overline{S_{n-1}} - \frac{n-2}{n} \times \overline{2n-1} \cdot \frac{2n-4}{2} \times \overline{S_{n-2}} + \frac{n-3}{n} \times \overline{2n-1} \cdot \frac{2n-2}{2} \cdot \frac{2n-6}{3} \overline{S_{n-3}} - \frac{n-4}{n} \times \overline{2n-1} \cdot \frac{2n-2}{2} \cdot \frac{2n-3}{3} \times \frac{2n-8}{4} \overline{S_{n-4}} + \&c.$ nearly, of which the general term is $\frac{n-l}{n} \times \overline{2n-1} \cdot \frac{2n-2}{3} \cdot \frac{2n-3}{3} \dots \frac{2n-l+1}{l-1} \times \frac{2n-2l}{l} \times \overline{S_{n-l}}$.

Ex. 5. Let the given series be of the formula $ax + bx^2 + cx^3 + \&c.$, and in it for x be substituted $p, -p, 2p, -2p, 3p, -3p, \dots, np, -np$ and mp , and for the sums of the resulting series be wrote respectively $S^1, S^{-1}, S^2, S^{-2}, S^3, S^{-3}, \dots S^n, S^{-n}$, and S_m ; then will $S_m =$

$$\frac{m \cdot \overline{m^2-1} \cdot \overline{m^2-4} \cdot \overline{m^2-9} \cdot \overline{m^2-16} \dots \overline{m^2-n-1}^2 \cdot \overline{m-n}}{n \cdot \overline{n^2-1} \cdot \overline{n^2-4} \cdot \overline{n^2-9} \cdot \overline{n^2-16} \dots \overline{n^2-n-1} \times 2n = 1 \cdot 2 \cdot 3 \cdot 4 \dots 2n} \\ \times S^{-n} + A \times \frac{m+n}{m-n} S^{+n} - \frac{2n}{1} \times B \times \frac{m-n}{m+n-1} S^{-n+1} - C \times \frac{m+n-1}{m-n-1} \times \\ S^{n-1} + \frac{2n-1}{2} \times D \times \frac{m-n-1}{m+n-2} \times S^{-n+2} + \frac{m+n-2}{m-n-2} \times E S^{n-2} - \frac{2n-2}{3} \times \\ F \times \frac{m-n-2}{m+n-3} S^{-n+3} - G \times \frac{m+n-3}{m-n-3} \times S^{n-3} + \&c. \text{ nearly, where}$$

the letters A, B, C, D, &c. respectively denote the preceding co-efficients. In general, the co-efficients of the terms S^{-n+s} and S^{n-s} will be respectively $M = \frac{2n-s+1}{s} \times L \times \frac{m-n-s+1}{m+n-s}$ and $M \times \frac{m+n-s}{m-n-s}$, where the letters L and M respectively denote their preceding co-efficients; the co-efficients are to be taken affirmatively, or negatively, according as s is an even or odd number.

Ex. 6. If for x in the preceding series be substituted $p, -p, 2p, -2p, 3p, -3p, \dots, n-1p, -n-1p, np$ respectively, then

S_m

$$Sm = \frac{m \cdot \overline{m-1} \cdot \overline{m^2-4} \cdot \overline{m^2-9} \cdot \overline{m^2-16} \dots (\overline{m^2-(n-1)^2})}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \dots (2n-1)} S^n - A \times$$

$$\frac{m-n}{m+n-1} \times S^{-n+1} - \frac{2n-1}{1} \times B \times \frac{m+n-1}{m-n-1} \times S^{n-1} + C \times \frac{m-n-1}{m+n-2} \times$$

$$S^{-n+2} + \frac{2n-2}{2} \times D \times \frac{m+n-2}{m-n-2} \times S^{n-2} - \&c. \text{ nearly, when } A, B, C,$$

&c. denote as before the preceding co-efficients. The coefficients of the terms S^{-n+1} and S^{n-1} will be respectively $L \times \frac{m-n-s+1}{m+n-s}$ and $\frac{2n-1}{s} \times M \times \frac{m+n-s}{m-n-s}$, L and M denoting the preceding co-efficients, which are to be taken negatively or affirmatively, as s is an even or an odd number. In this series when $x=0$, the correspondent sum $=0$.

Ex. 7. Let the given series be of the formula $a + bx + cx^2 + dx^3 + \&c.$; and in it for x be substituted $0, p, -p, 2p, -2p, 3p, -3p \dots np, -pp$ and mp , and for the sums of the resulting series be wrote as before $S^0, S^1, S^{-1}, S^2, S^{-2}, \dots S^n, S^{-n}$, and S^m ; then will $Sm = \frac{m \cdot \overline{m^2-1} \cdot \overline{m^2-4} \cdot \overline{m^2-9} \dots (\overline{m^2-n-1^2}) \times \overline{m-n}}{n \cdot \overline{n^2-1} \cdot \overline{n^2-4} \cdot \overline{n^2-9} \dots (\overline{n^2-n-1^2}) \times 2n}$
 $\times S^{-n} + A \times \frac{m+n}{m-n} \times S^{+n} - \frac{2n}{1} \times B \times \frac{m-n}{m+n-1} S^{-n+1} - \&c.$ this series observes the same law as the series given in Ex. 5. and only differs from it by the last term S^0 not vanishing, that is, being $=0$.

Ex. 8. Let the series be of the preceding formula $a + bx + cx^2 + dx^3 + \&c.$, and in it for x be substituted $0; p, -p; 2p, -2p; 3p, -3p; \dots n-1p, -n-1p, np$, and mp , and the sums resulting be $S^0, S^1, S^{-1}, S^2, S^{-2}, \dots S^{n-1}, S^{-n+1}, S^n$ and S^m ; then will $Sm = \frac{m \cdot \overline{m^2-1} \cdot \overline{m^2-4} \dots \overline{m^2-(n-1)^2}}{1 \cdot 2 \cdot 3 \cdot 4 \dots 2n-1} S^n - A \times \frac{m-n}{m+n-1} \times$
 $S^{-n+1} - \&c.$ the same series as in Ex. 6. and differs from it only by the last term S^0 not vanishing.

Ex. 9. Let the series be of the same formula $a + bx + cx^2 + dx^3 + \&c.$ and in it for x be substituted $p, -p, 3p, -3p, 5p, -5p, 7p, -7p, \dots np, -np$ and mp ; and the sums resulting be $S^1, S^{-1}, S^3, S^{-3}, S^5, S^{-5}, S^7, S^{-7}, \dots S^n, S^{-n}$, and S^m ; then will $S^m =$

$$\frac{m^2 - 1 \cdot m^2 - 9 \cdot m^2 - 25 \cdot m^2 - 49 \dots m^2 - n - 2^2 \times m + n}{n^2 - 1 \cdot n^2 - 9 \cdot n^2 - 25 \cdot n^2 - 49 \dots n^2 - n - 2^2 \times 2n = 2^n \times 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \dots n} \\ \times S^n - A \times \frac{m-n}{m+n} \times S^{-n} - \frac{n}{1} \times B \times \frac{m+n}{m-n+2} \times S^{n-2} + C \times \frac{m-n+2}{m+n-2} \times \\ S^{-n+2} + \frac{n-1}{2} \times D \times \frac{m+n-2}{m-n+4} \times S^{n-4} - E \times \frac{m-n+4}{m+n-4} \times S^{-n+4} - \frac{n-2}{3} F \\ \times \frac{m+n-4}{m-n+6} \times S^{n-6} + \frac{m-n+6}{m+n-6} \times G \times S^{-n+6} + \frac{n-3}{4} \times H \times \frac{m+n-6}{m-n+8} \times \\ S^{-n+8} - \&c. \text{ nearly, where the letters } A, B, C, D, E, \&c. \text{ denote the preceding co-efficients of the terms } S^n, S^{-n}, S^{n-2}, S^{-n+2}, S^{n-4}, S^{-n+4}, S^{n-6}, \&c. \text{ respectively. The co-efficients of the terms } S^{2n-2s} \text{ and } S^{-n+2s} \text{ will be } M = \frac{n-s+1}{s} \times L \times$$

$\frac{m+n-2s+2}{m-n+2s}$ and $N = M \times \frac{m-n+2s}{m+n-2s}$; where L, M , and N denote the co-efficients of the terms immediately preceding each other, that is, of the terms $S^{-n+2s-2}, S^{n-2s}$, and S^{-n+2s} . The sign of the first co-efficient M will be $+$ or $-$, according as s is even or odd; the second term N will have a contrary sign to the first.

These series may be made to begin from any term, which may be easily found by the method of correspondent values, and the subsequent terms from it by the given law; its preceding terms may be deduced from the same law reversed, that is, by putting the numerators of the fractions multiplied into it for the denominators, and the denominators for the numerators.

From these different serieses may be formed, by adding two or more terms of the given series together for a term of the required series; which method has been applied to converging series in general in the Meditations.

13. The method of correspondent values easily affords a resolution of the problems contained in Mr. BRIGG's or Sir ISAAC NEWTON's method of differences.

Ex. 1. Let the quantity be of the formula $a + bx + cx^2 + dx^3 + \&c. \dots x^n = y$, and $n + 1$ correspondent values of x and y be given, viz. $p, q, r, s, \&c.$ of x ; $Sp, Sq, Sr, Ss, \&c.$ of y ; then will $y = \frac{x-p \cdot x-q \cdot x-r \cdot x-s \cdot \&c.}{p-q \cdot p-r \cdot p-s \cdot \&c.} \times Sp + \frac{x-p \cdot x-q \cdot x-r \cdot x-s \cdot \&c.}{q-p \cdot q-r \cdot q-s \cdot \&c.} \times Sq + \frac{x-p \cdot x-q \cdot x-r \cdot x-s \cdot \&c.}{r-p \cdot r-q \cdot r-s \cdot \&c.} \times Sr + \frac{x-p \cdot x-q \cdot x-r \cdot x-s \cdot \&c.}{s-p \cdot s-q \cdot s-r \cdot \&c.} \times Ss + \&c.$

The truth of this problem very easily appears by writing $p, q, r, s, \&c.$ for x in the given series.

All the preceding examples may be applied to this case, by writing x for m in the given series; hence the resolutions of several cases of equi-distant ordinates by easy and not inelegant serieses, amongst which are included the two cases commonly given on this subject.

14. If a quantity be required, which proceeds according to the dimensions of x , reduce the above given value of y into a quantity proceeding according to the dimensions of x , and there results $y = \left(\frac{Sp}{p-q \cdot p-r \cdot p-s \cdot \&c. = A} + \frac{Sq}{q-p \cdot q-r \cdot q-s \cdot \&c. = B} + \frac{Sr}{r-p \cdot r-q \cdot r-s \cdot \&c. = C} + \frac{Ss}{s-p \cdot s-q \cdot s-r \cdot \&c. = D} + \&c. \right) \times x^n -$

$$\left(\frac{Sp \times q + r + s + \&c.}{A} + \frac{Sq \times p + r + s + \&c.}{B} + \frac{Sr \times p + q + s + \&c.}{C} + \frac{Ss \times p + q + r + \&c.}{D} + \&c. \right) x^{n-1} + \left(\frac{Sp \times qr + qs + rs + \&c.}{A} + \right.$$

$$\frac{Sq \times \overline{pr} + \overline{ps} + \overline{rs} + \&c.}{B} + \frac{Sr \times \overline{pq} + \overline{ps} + \overline{qs} + \&c.}{C} + \frac{Ss \times \overline{pq} + \overline{pr} + \overline{qr} + \&c.}{D} + \&c.) \\ \times x^{n-2} - \left(\frac{Sp \times \overline{qrs} + \&c.}{A} + \frac{Sq \times \overline{prs} + \&c.}{B} + \frac{Sr \times \overline{pqs} + \&c.}{C} + \frac{Ss \times \overline{pqr} + \&c.}{D} + \right. \\ \left. \&c. \right) x^{n-3} + \&c.$$

The law and continuation of this series is evident to any one versant in these matters from inspection.

These fractions may be reduced to a common denominator by substituting for Sp and A the products $Sp \times P$ and $A \times P$, where $P = \overline{q-r} \cdot \overline{q-s} \cdot \overline{r-s} \cdot \&c.$; for Sq and B the products $Sq \times Q$ and $B \times Q$, where $Q = \overline{p-r} \cdot \overline{p-s} \cdot \overline{r-s} \cdot \&c.$; for Sr and C the products $Sr \times R$ and $C \times R$, where $R = \overline{p-q} \cdot \overline{p-s} \cdot \overline{q-s} \cdot \&c.$; for Ss and D the products $Ss \times S'$ and $C \times S'$, where $S' = \overline{p-q} \cdot \overline{p-r} \cdot \overline{q-r} \cdot \&c. \&c.$

The fractions, in particular cases, will often be reducible to lower terms.

15. Let $y = ax^b + bx^{b+1} + cx^{b+2} + \&c.$, and the correspondent values of x and y be given as before, then will $y =$

$$\frac{x^b \times \overline{x^l - q^l} \times \overline{x^l - r^l} \times \overline{x^l - s^l} \times \&c.}{p^b \times \overline{p^l - q^l} \times \overline{p^l - r^l} \times \overline{p^l - s^l} \times \&c.} \times Sp + \frac{x^b \times \overline{x^l - p^l} \times \overline{x^l - r^l} \times \overline{x^l - s^l} \times \&c.}{q^b \times \overline{q^l - p^l} \times \overline{q^l - r^l} \times \overline{q^l - s^l} \times \&c.} \times Sq \\ + \frac{x^b \times \overline{x^l - p^l} \times \overline{x^l - q^l} \times \overline{x^l - s^l} \times \&c.}{r^b \times \overline{r^l - p^l} \times \overline{r^l - q^l} \times \overline{r^l - s^l} \times \&c.} \times Sr + \frac{x^b \times \overline{x^l - p^l} \times \overline{x^l - q^l} \times \overline{x^l - r^l} \times \&c.}{s^b \times \overline{s^l - p^l} \times \overline{s^l - q^l} \times \overline{s^l - r^l} \times \&c.} \times Ss + \&c.$$

This series may in the same manner as the preceding be reduced to terms, proceeding according to the dimensions of x ; and the serieses given in the examples may (*mutatis mutandis*) be predicated of it.

16. A more general method of correspondent values is given in the *Meditationes*, as also the subsequent $y = \frac{\overline{x-q} \cdot \overline{x-r} \cdot \overline{x-s} \cdot \&c.}{p-q \cdot p-r \cdot p-s \cdot \&c.}$

$$\begin{aligned} & \times Sp + \frac{\overline{x-p} \cdot \overline{x-r} \cdot \overline{x-s} \cdot \&c.}{q-p \cdot q-r \cdot q-s \cdot \&c.} \times Sq + \frac{\overline{x-p} \cdot \overline{x-q} \cdot \overline{x-s} \cdot \&c.}{r-p \cdot r-q \cdot r-s \cdot \&c.} \times Sr + \\ & \&c. \text{ as in Ex. 1. } = Sp + (x-p) \left(\frac{1}{p-q} \times Sp + \frac{1}{q-p} \times Sq \right) + (x-p) \\ & (x-q) \left(\frac{1}{p-q} \times \frac{1}{p-r} \times Sp + \frac{1}{q-p} \times \frac{1}{q-r} \times Sq + \frac{1}{r-p} \times \frac{1}{r-q} \times Sr \right) + \\ & (x-p)(x-q)(x-r) \left(\frac{1}{p-q} \cdot \frac{1}{p-r} \cdot \frac{1}{p-s} \cdot \times Sp + \frac{1}{q-p} \cdot \frac{1}{q-r} \cdot \frac{1}{q-s} \right. \\ & \left. \times Sq + \frac{1}{r-p} \cdot \frac{1}{r-q} \cdot \frac{1}{r-s} \times Sr + \frac{1}{s-p} \cdot \frac{1}{s-q} \cdot \frac{1}{s-r} \times Ss \right) - \&c. \end{aligned}$$

The equality of these two different quantities will easily appear by finding the co-efficients of both, which are multiplied into the same given value of y as $Sp, Sq, Sr, \&c.$ and the same power of x ; for with very little difficulty they will in general be found equal.

It is evident from this resolution that, giving the ordinates and their respective distances from each other, the value of any other ordinate at a given distance from the preceding, found by this method, will result the same, whatever may be the point assumed from which the absciss is made to begin.

3.

1. Let a series be $Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$ of such a formula that if in it for x be substituted $a+b$, there results a series $A \times \overline{a+b} + B \times \overline{a+b}^2 + C \times \overline{a+b}^3 + D \times \overline{a+b}^4 + \&c. = (Aa + Ba^2 + Ca^3 + Da^4 + \&c.) \times (1 + qb + rb^2 + sb^3 + tb^4 + \&c.) + (1 + qa + ra^2 + sa^3 + ta^4 + \&c.) \times (Ab + Bb^2 + Cb^3 + Db^4 + \&c.)$ then will the series $Ax + Bx^2 + Cx^3 + Dx^4 + \&c. = Ax + \frac{2B}{1 \cdot 2}x^2 + \frac{2 \cdot 3C}{1 \cdot 2 \cdot 3}x^3 + \frac{24ABC - 8B^3}{1 \cdot 2 \cdot 3 \cdot 4A^2}x^4 + \frac{36C^2A^2 + 24ACB^2 - 16B^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5A^3}x^5 + \frac{9 \cdot 24A^2BC^2 - 4 \times 24AB^3C}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6A^4}x^6 + \frac{216C^3A^3 + 432A^2B^2C^2 - 384ACB^4 + 64B^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7A^5}x^7 + \&c. ;$

$$+ \&c.; \text{ and the series } 1 + qx + rx^2 + sx^3 + tx^4 + \&c. = 1 + \frac{B}{A}x + \frac{6CA - 2B^2}{1 \cdot 2A^2}x^2 + \frac{18CAB - 8B^3}{1 \cdot 2 \cdot 3A^3}x^3 + \frac{36C^2A^2 - 8B^4}{1 \cdot 2 \cdot 3 \cdot 4A^4}x^4 + \frac{180C^2A^2B - 120ACB^3 + 16B^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5A^5}x^5 + \frac{216C^3A^3 + 216A^2C^2B^2 - 288ACB^4 + 64B^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6A^6}x^6 + \&c.$$

The terms of these two series can easily be deduced by the subsequent method. Let $Kx^{n-2} + L^{n-1} + Mx^n$, be successive terms of the series $Ax + Bx^2 + Cx^3 + \&c.$, and $K^1x^{n-2} + L^1x^{n-1}$ successive terms of the series $1 + qx + rx^2 + sx^3 + tx^4 + \&c.$; then will $M = \frac{2A^2 \times B \times K^1 + 6CAK - 2B^2K}{n \cdot n - 1 \times A^2}$ and $L^1 = \frac{n \times A \times M - B \times xL}{A^2}$.

Cor. 1. Let $B = 0$, and the two series $Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$ and $1 + qx + rx^2 + \&c.$ become respectively $Ax + \frac{2 \cdot 3}{2 \cdot 3}Cx^3 + \frac{2^2 \cdot 3^2}{2 \cdot 3 \cdot 4 \cdot 5} \times \frac{C^2}{A}x^5 + \frac{2^3 \cdot 3^3}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \times \frac{C^3}{A^2}x^7 + \frac{2^4 \cdot 3^4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} \times \frac{C^4}{A^3}x^9 + \&c.$, and $1 + \frac{2 \cdot 3}{1 \cdot 2} \times \frac{C}{A}x^2 + \frac{2^2 \cdot 3^2}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{C^2}{A^2}x^4 + \frac{2^3 \cdot 3^3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \times \frac{C^3}{A^3}x^6 + \&c.$

If in these series for A be substituted 1, and for C be substituted $-\frac{1}{2 \cdot 3}$, there will result the series $x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} - \&c.$, and $1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \&c.$ which give the sine and cosine in terms of the arc x .

Cor. 2. Let $C = 0$, and the above-mentioned series $Ax + Bx^2 + \&c.$ becomes $Ax + \frac{2}{1 \cdot 2}Bx^2 * - \frac{2^3}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{B^3}{A^2}x^4 - \frac{2^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \times \frac{B^4}{A^3}x^5 * + \frac{2^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \times \frac{B^6}{A^5}x^7 + \frac{2^7}{1 \cdot 2 \cdot 3 \cdot 4 \dots 8} \times \frac{B^7}{A^6}x^8 * - \frac{2^9}{1 \cdot 2 \cdot 3 \cdot 4 \dots 10} \times \frac{B^9}{A^8}x^{10} - \frac{2^{10}}{1 \cdot 2 \cdot 3 \dots 11} \times \frac{B^{10}}{A^9}x^{11} + \&c.$ The law of

this

this series is, first, that every third term vanishes; and, secondly, the signs of every two successive terms change alternately from + to - and - to +; and, lastly, the co-efficient of the term x^n is $\frac{2^{n-1}}{1 \cdot 2 \cdot 3 \dots n} \times \frac{B^{n-1}}{A^{n-2}}$; and the series $1 + qx + rx^2 + \&c.$ becomes $1 + \frac{B}{A}x - \frac{2B^2}{1 \cdot 2A^2}x^2 - \frac{2^3B^3}{1 \cdot 2 \cdot 3A^3}x^3 - \frac{2^3B^4}{1 \cdot 2 \cdot 3 \cdot 4A^4}x^4 + \frac{2^4B^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5A^5}x^5 + \frac{2^6B^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6A^6}x^6 + \frac{2^6 \times B^7}{1 \cdot 2 \cdot 3 \dots 7A^7}x^7 - \&c.$ In this series the signs of three successive terms alternately change from + to - and - to +; and the co-efficient of the term x^n is $\frac{2^n \times B^n}{1 \cdot 2 \cdot 3 \dots nA^n}$ or $\frac{2^{n-1} \times B^n}{1 \cdot 2 \cdot 3 \dots nA^n}$ according as n is divisible by 3 or not.

2. Let a series $1 + Px + Qx^2 + Rx^3 + Sx^4 + Tx^5 + \&c.$ be of such a formula, that if in it for x be substituted $a+b$, there results a series $1 + P \times \overline{a+b} + Q \times \overline{a+b}^2 + R \times \overline{a+b}^3 + S \times \overline{a+b}^4 + \&c. = (1 + Pa + Qa^2 + Ra^3 + Sa^4 + \&c.) \times (1 + Pb + Qb^2 + Rb^3 + Sb^4 + \&c.) + (Aa + Ba^2 + Ca^3 + Da^4 + \&c.) \times (Ab + Bb^2 + Cb^3 + Db^4 + \&c.)$, then will the series $Ax + Bx^2 + Cx^3 + Dx^4 + \&c. = Ax + Bx^2 + \left(\frac{2B^2}{3A} - \frac{PB}{3} + A \times \frac{A^2 + P^2}{6}\right)x^3 + \frac{2B^3 - 2PAB^2 + A^2 \times \overline{A^2 + P^2} \times B}{6A^2}x^4 + \&c.$, and the series $1 + Px + Qx^2 + Rx^3 + \&c. = 1 + Px + \frac{A^2 + P^2}{2}x^2 + \frac{2AB + P \times \overline{A^2 + P^2}}{6}x^3 + \frac{4B^2 + \overline{A^2 + P^2}^2}{24}x^4 + \&c.$ Let $Kx^{n-2} + Lx^{n-1} + Mx^n$ be successive terms of the series $Ax + Bx^2 + Cx^3 + \&c.$, and $K'x^{n-2} + L'x^{n-1} + M'x^n$ successive terms of the series $1 + Px + Qx^2 + Rx^3 + \&c.$; then will $A \times L + P \times L' = n \times M'$ and $B \times K + Q \times K' = n \cdot \frac{n-1}{2} \times M'$ expresses the law of the serieses.

Cor. Let $B=0$, then the series $Ax+Bx^2+Cx^3+Dx^4=A \times (x+\frac{P^2 \times A^2}{2 \cdot 3}x^3+\frac{(P^2+A^2)^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}x^5+\frac{(P^2+A^2)^3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}x^7+\&c.)$, and the series $1+Px+Qx^2+Rx^3+\&c.=1+Px+\frac{P^2+A^2}{1 \cdot 2}x^2+P \times \frac{P^2+A^2}{1 \cdot 2 \cdot 3}x^3+\frac{(P^2+A^2)^2}{1 \cdot 2 \cdot 3 \cdot 4}x^4+P \times \frac{(P^2+A^2)^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}x^5+\frac{(P^2+A^2)^3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}x^6+\&c.$; the co-efficient of the term x^n will be $(P^2+A^2)^{\frac{n}{2}}$ or $P \times (P^2+A^2)^{\frac{n-1}{2}}$, according as n is even or odd.

If in the equations before given for x be substituted $a=b$ instead of $a+b$, then in the other quantities for b substitute $-b$.

3. If in Case 2. the difference between the two quantities $(1+Pa+Qa^2+\&c.) \times (1+Pb+Qb^2+\&c.)$ and $(Aa+Ba^2+Ca^2+\&c.) \times (Ab+Bb^2+Cb^2+\&c.)$ is assumed $=1+P \times \overline{a+b}+Q \times \overline{a+b^2}+\&c.$, then in the serieses before given for $A, B, C, \&c.$ write respectively $\sqrt{-1}A, \sqrt{-1}B, \sqrt{-1}C, \&c.$, and there will result the corresponding serieses.

The same principles may be applied to many other cases.

4. Equations of these formulæ may be useful, when the sums of the serieses correspondent to a value (a) of x are given, and the sums of the series correspondent to a value ($a+b$) of x is required, b having a small ratio to a : for instance, let the given series be $x-\frac{x^3}{2 \cdot 3}+\frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5}-\frac{x^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}+\&c.$; the equation found in the first case is $a+b-\frac{(a+b)^3}{2 \cdot 3}+\frac{(a+b)^5}{2 \cdot 3 \cdot 4 \cdot 5}-\&c.=(a-\frac{a^3}{2 \cdot 3}+\frac{a^5}{2 \cdot 3 \cdot 4 \cdot 5}-\&c.) \times (1-\frac{b^2}{1 \cdot 2}+\frac{b^4}{1 \cdot 2 \cdot 3 \cdot 4}-\&c.)+(1-\frac{a^2}{1 \cdot 2}+\frac{a^4}{1 \cdot 2 \cdot 3 \cdot 4}-\&c.) \times (b-\frac{b^3}{2 \cdot 3}+\frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5}-\&c.);$
but

but $a - \frac{a^3}{2 \cdot 3} + \frac{a^5}{2 \cdot 3 \cdot 4 \cdot 5} - \&c.$, and $1 - \frac{a^2}{1 \cdot 2} + \frac{a^4}{2 \cdot 3 \cdot 4} - \&c.$ are the sine (s) and cosine (c) of an arc a of a circle whose radius is 1; and, consequently, if the sine s and cosine c of an arc a be given, the sine of an arc $(a+b) = s \times (1 - \frac{b^2}{2} + \frac{b^4}{24} - \&c.) + c(b - \frac{b^3}{2 \cdot 3} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} - \&c.)$, which series, if b be very small in proportion to a , converges much faster than the common series for finding the sine from the arc: it has been given from different principles in the *Meditationes*, and is also easily deducible from the series for finding the sine and cosine from the arc by the propositions usually given in plane trigonometry: the cosine of the same arc $(a+b) = c \times (1 - \frac{b^2}{1 \cdot 2} + \frac{b^4}{2 \cdot 3 \cdot 4} - \&c.) - s \times (b - \frac{b^3}{1 \cdot 2 \cdot 3} + \frac{b^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \&c.)$,

Ex. 2. Let the series be $\overline{a+b} + \frac{\overline{a+b}^3}{2 \cdot 3} + \frac{\overline{a+b}^5}{2 \cdot 3 \cdot 4 \cdot 5} + \&c. =$
 $(a + \frac{a^3}{2 \cdot 3} + \frac{a^5}{2 \cdot 3 \cdot 4 \cdot 5} + \&c.) \times (1 + \frac{b^2}{1 \cdot 2} + \frac{b^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c.) +$
 $(1 + \frac{a^2}{1 \cdot 2} + \frac{a^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c.) \times (b + \frac{b^3}{1 \cdot 2 \cdot 3} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \&c.)$;
 but $a + \frac{a^3}{1 \cdot 2 \cdot 3} + \&c. = x$, and $1 + \frac{a^2}{1 \cdot 2} + \frac{a^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c. =$
 $\sqrt{1+x^2}$, if a be the hyperbolic log. of $x + \sqrt{1+x^2}$; therefore
 $a + b + \frac{\overline{a+b}^3}{2 \cdot 3} + \frac{\overline{a+b}^5}{2 \cdot 3 \cdot 4 \cdot 5} + \&c. = x \times (1 + \frac{b^2}{2} + \frac{b^4}{2 \cdot 3 \cdot 4} + \&c.) +$
 $\sqrt{1+x^2} \times (b + \frac{b^3}{2 \cdot 3} + \&c.)$

Let $b + \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4 \cdot 5} + \&c. = y$, and $(x + \sqrt{1+x^2} \times$
 $(y + \sqrt{1+y^2}) = V$, then will $\overline{a+b} + \frac{\overline{a+b}^3}{2 \cdot 3} + \frac{\overline{a+b}^5}{2 \cdot 3 \cdot 4 \cdot 5} + \&c. =$
 $\frac{x}{2} V - \frac{1}{V}$.

5. Let a quantity P be a function of x , or the fluent of a function of $x \times \dot{x}$, and the value X of it when $x=a$ be known, and the value of it when $x=a+b$ be required. Find a series of which the first term is X , and which proceeds according to the dimensions of b , if b be a very small quantity, and in general at least so small that the series from $x=a$ to $x=a+b$ neither becomes infinite or 0.

In the same manner, if an algebraical or fluxional equation or equations, expressing the relations between x, y, z, v , &c. be given, find the correspondent values of y, z, v , &c. to $x=a$, which let be Y, Z, V , &c.; then find serieses for y, z, v , &c. of which the first terms let be Y, Z, V , &c. respectively, and which proceed according to the dimensions of b , but subject to the same conditions as in the preceding case.

From fluxional equations may be deduced series which express the value of y , &c. in terms of x , and always diverge, or always converge, whatever may be its value, as appears from the *Meditationes*.

